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Dynamic Systems with Random Initial State

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SUMMARY

A perturbation scheme is described to treat time variable cryptodeterministic systems. According to Moyal's degree of randomness criteria the method provides a complete stochastic characterization of the system response. Certain digital computational features, when the perturbation scheme is not applicable, are also outlined. For an assumed random initial state, the results are then applied to describe the transient flapping oscillations of a helicopter blade which in forward flight has periodically varying aerodynamic damping and spring parameters.

1. Introduction

In the simulation of physical systems in a stochastic environment, randomness in the idealized discrete system is introduced through several sources – initial state, inputs, system parameters and eigenvalues. In recent years such stochastic systems have received considerable attention [1–15]. In the general field of stochastic differential equations, Syski [1] has reviewed the state of knowledge and significant advances prior to 1967. Boyce [2] has given a comprehensive account of random eigenvalue methods proposed up to 1968. From the viewpoint of engineering applications, Hasselmann [3] has treated constant parameter systems with random initial state and with stationary random inputs, and Van Trees [4] has given a computationally convenient formulation to compute the variance matrix of time variable systems with random initial state and white noise excitations. Quantitative response description of systems with periodically varying parameters and separable nonstationary excitations are also given in references 5 and 6. As a generalization to Van Trees' formulation Anderson *et al.* [7] have proposed an algorithm, solution of a matrix Riccati equation, to design variable shaping filters for nonstationary random inputs. With reference to structural applications, Collins and Thomson [8] have studied random eigenvalue matrices and Hoshiya [9] has developed a perturbation method to treat systems with random eigenvalues and random parameters. Closed form solutions are also given by Barnoski and Maurer [10], and Sveshnikov [11] for the response variance of constant parameter systems with exponential type separable nonstationary excitations.

Especially for systems with random eigenvalues or random parameters the so-called honest methods [2, 12] and hierarchy techniques [2, 13, 14] have been studied in some detail. As the state transition matrix of variable systems is rarely available in closed form the applicability of honest methods is very much limited. In hierarchy techniques heuristic assumptions are often made on the partial or complete independence of different random quantities so that certain higher moments can be expressed in terms of first and second order moments.

Most of these above methods describe the system state and eigenvalues within the framework of the correlation or second moment theory, which in general, renders an incomplete stochastic characterization. This situation is not surprising for the fact that systems with random parameters, except in certain special cases, require the construction of stochastic weighting functions [15] whose computational features are still in an initial stage. As an additional complexity, even in linear stochastic equations, the relation between the random parameters and random eigenvalues or response is nonlinear. Even for systems with random elements only in the inputs, the evaluation of higher order moments involves considerable computational effort.

Systems where only the initial states are random are cryptodeterministic in the sense that the state follows deterministic laws and with the increase in time from the origin the stochastic content in the response decreases. Therefore from the consideration of stochastic process theory such systems are one of the simplest models of stochastic differential equations. In principle the joint density of the initial state and the Jacobian of the coordinate transformations should characterize the final state. But the system being cryptodeterministic, the range and the extent to which the response is stochastic is also important. It is here the concept of degree of randomness introduced by Moyal [1, 16, 17] is helpful.

This paper is concerned with such cryptodeterministic systems with time variable parameters. Such systems are frequently encountered in engineering and merit further study. A flight structure approaching clear air region immediately after passing through low terrain turbulence is one such example. A perturbation scheme is described for the complete stochastic characterization of the response. Treated for illustrative purposes are the transient flapping (out of plane) oscillations of a lifting rotor blade which in forward flight represents a time variable system.

2. Problem Description

Consider a linear system with n degrees of freedom and with the random initial state $X(t_0)$. Then the state equation can be set in the form

$$\dot{X}(t) = F(t)X(t), \qquad t_0 < t \leqq T \tag{1}$$

or

$$\dot{x}_i(t) = f_{ij}(t)x_j(t); \qquad i, j = 1, 2, \ldots, 2n$$

For simplicity it is stipulated that the state vector is identical to the response vector and the mean value of the initial state is zero. With the solution of the state transition matrix equation

$$\dot{\Phi}(t, \tau) = F(t)\Phi(t, \tau), \qquad \Phi(\tau, \tau) = I_{2n} \tag{2}$$

the system state for a given initial state $X(t_0)$ can be expressed as

$$X(t) = \Phi(t, t_0)X(t_0) \tag{3a}$$

An important property which will be applied subsequently is the transition property of $\Phi(t, \tau)$:

$$\Phi(t_1, t_2)\Phi(t_2, t_3) = \Phi(t_1, t_3) \quad \text{for all} \quad t_1, t_2 \text{ and } t_3 \tag{3b}$$

With the initial state being described in terms of its correlation matrix

$$E[X(t_0)X^T(t_0)] = \overset{*}{P}_0, \tag{4}$$

the state variance and the state correlation matrices can be expressed as [4, 7]

$$P(t) = R_x(t, t) = \Phi(t, t_0)\overset{*}{P}_0\Phi^T(t, t_0) \tag{5}$$

$$E[X(t_1)X(t_2)^T] = R_x(t_1, t_2) = \begin{cases} \Phi(t_1, t_2)P(t_2) & \text{for } t_1 \geqq t_2 \\ P(t_1)\Phi^T(t_2, t_1) & \text{for } t_1 \leqq t_2 \end{cases} \tag{6}$$

Without actually computing the state transition matrix one can also use the relation [4]

$$\dot{P}(t) = F(t)P(t) + P(t)F^T(t), \quad P(t_0) = \overset{*}{P}_0 \tag{7}$$

to evaluate $P(t)$ directly.

Let the state matrix $F(t)$ be close to constant matrix F_0 such that

$$F(t) = F_0 + \varepsilon F_1(t) \tag{8a}$$

where $F_1(t)$ is the small time variable part. In other words the elements of F_0 substantially dominate over the corresponding elements of $F_1(t)$, that is

$$|f_{0,ij}| \gg |f_{1,ij}(t)| \ ; \qquad i, j = 1, 2, \ldots, 2n \tag{8b}$$

Now the solution of equation (7) up to first order perturbation correction can be expressed in the form [18]

$$P(t) = P_0(t) + \varepsilon P_1(t) \tag{8c}$$

The substitution of equations (8a) and (8c) in (7), with the usual perturbation procedure, gives the zero-order equation

$$\dot{P}_0(t) = F_0 P_0(t) + P_0(t) F_0^T \ , \qquad P(t_0) = \overset{*}{P}_0 \tag{9a}$$

and the first-order correction equation

$$\dot{P}_1(t) = F_0 P_1(t) + P_1(t) F_0^T + (F_1(t) P_0(t) + P_0(t) F_1^T(t)) \ , \quad P_1(t_0) = 0 \tag{9b}$$

Higher order perturbation corrections can be similarly expressed by maintaining ε^2, ε^3 etc. in the perturbation expansion. Using the perturbed state matrix, equation (8a), one can also compute the state transition matrix [18] or the response correlation matrix [5].

However the description in terms of the first two moments of the response process is in general incomplete. For a stronger stochastic characterization consider equation (3a) with a joint density function of the initial state

$$f_0(X(t_0)) = f_0(x_1(t_0) \ , \quad x_2(t_0), \ldots, x_{2n}(t_0)) \tag{10}$$

As the transformation from the initial state $X(t_0)$ to the final state $X(t)$ is uniquely governed by the state transition matrix, the joint density of $X(t)$ can be expressed as [1]

$$f(X(t) = f(x_1(t) \ , \quad x_2(t), \ldots, x_{2n}(t)) = f_0(X(t)) \frac{1}{|J(t)|} \tag{11}$$

where

$$J(t) = \exp \int_{t_0}^{t} (\text{trace } F(\theta)) d\theta \tag{12a}$$

Note that the use of (11) implies that the components of $X(t_0)$ are expressed in terms of the components of the final state. For this purpose, using the transition property, equation (3b), one gets

$$X(t_0) = \Phi^{-1}(t, t_0) X(t) = \Phi(t_0, t) X(t) \tag{12b}$$

With deterministic inputs one has to include the contributions from the superposition integral whose integrand is the product of the state transition matrix and the input vector. For simplicity only zero-input scalar response is treated.

Similarly in linear combination with the $2n$ components of the initial state, the response component, say $x_1(t)$, can be expressed at $2n$ different t values:

$$x_1(t_i) = \sum_{j=1}^{2n} \Phi_{ij}(t_i) x_j(t_0) \ ; \qquad i = 1, 2, \ldots, 2n, \ t_0 < t \leqq T \tag{13}$$

where $\phi_{11}(t)$, $\phi_{12}(t)$, ..., $\phi_{1,2n}(t)$ are the $2n$ fundamental set of solutions which are normally the elements in the first row of the state transition matrix. When the $2n$ equations in (13) are independent (certain degenerate cases will be discussed subsequently) the joint density function for random variables $x_1(t_1)$, $x_1(t_2)$, ..., $x_1(t_{2n})$ can be expressed as

$$f(x_1(t_1), x_1(t_2), \ldots, x_1(t_{2n})) = f_0(X(t_0)) \cdot \frac{1}{|J(t_1, t_2, \ldots, t_{2n})|} \tag{14}$$

where

$$J(t_1, t_2, \ldots, t_{2n}) = \begin{vmatrix} \phi_{11}(t_1) & \phi_{12}(t_1) & \cdots & \phi_{1,2n}(t_1) \\ \vdots & \vdots & & \vdots \\ \phi_{11}(t_{2n}) & \phi_{12}(t_{2n}) & \cdots & \phi_{1,2n}(t_{2n}) \end{vmatrix} \tag{15}$$

Viewed geometrically in a $2n$ dimensional space, the Jacobian in (15) serves dual purposes. First, one can ascertain whether certain pairs of random variables generated according to (13) are perfectly correlated. Even within the transient region for some specific combination of points on the time axis the joint density of the response random variables could contain singularity line masses. Second, one can estimate the range after which the response process is essentially deterministic. Because, for stable systems with bounded inputs inducing bounded responses, the influence of the initial state decreases with the growth of the response.

Concerning the stochastic structure of the response, observe that higher the order of the probability density function stronger is the stochastic characterization. Therefore higher the dimension of the state, less deterministic the transients are. At this stage one uses the concept of degree of randomness introduced by Moyal [2, 16, 17] in statistical mechanics. A random process $\{x, (t)\}$ has $2n$ degrees of randomness if $2n$ is the smallest integer when the Jacobian in (15) does not vanish *identically* for all values of $t_1, t_2, \ldots, t_{2n}$. That is, $f(t_1, t_2, \ldots, t_k)$ is singular for $k > 2n$, leading to line masses and $f(t_1, t_2, \ldots, t_{2n})$ is *not necessarily singular* for $k < 2n$ points. If $J(t_1, t_2, \ldots, t_{2n})$ does not vanish for any finite values of k the process $\{x_1(t)\}$ is said to be completely random. Thus, if $x_1(t)$ is known at $2n$ distinct points, say at $t_1 < t_2 < \ldots < t_{2n}$, then $x_1(t_k)$ values for $k > 2n$ can be determined in terms of these $x_1(t_1), x_2(t_2), \ldots x_2(t_{2n})$ values. If $x_1(t)$ is known only at $k < 2n$ points, then the remaining $2n-k$ values are subject to chance mechanism, hence the response description is incomplete. These three cases, $k > 2n$, $k = 2n$ and $k < 2n$ can be quantitatively stated using multivariate dirac delta functions [19]. Henceforth it will be stipulated that $k = 2n$. Thus according to Moyal's degree of randomness criteria the transient response of a cryptodeterministic system with n degrees of freedom will have $2n$ degrees of randomness and is completely described by (11), (12a), (14) and (15).

To exemplify consider a single degree of freedom system with the state matrix

$$F_0 = \begin{vmatrix} 0 & 1 \\ -\omega_0^2 & -2\zeta\omega_0 \end{vmatrix}, \quad 0 < t \leqq T \tag{16a}$$

which has the state transition matrix with elements

$$\alpha_{11} = (\exp(-\zeta\omega_0 t))\left(\cos\bar{\omega}t + \frac{\zeta\omega_0}{\bar{\omega}}\sin\bar{\omega}t\right) \tag{16b}$$

$$\alpha_{12} = (1/\bar{\omega})\sin\bar{\omega}t(\exp(-\zeta\omega_0 t)) \tag{16c}$$

$$\alpha_{21} = \dot{\alpha}_{11} \quad \text{and} \quad \alpha_{22} = \dot{\alpha}_{12}$$

where $\bar{\omega} = \omega_0\sqrt{1-\zeta^2}$.

With the response defined by the relation

$$x_1(t) = x_1(0)\alpha_{11}(t) + \dot{x}_1(0)\alpha_{12}(t) \tag{16d}$$

the Jacobian, equation (15), simplifies to

$$J(t_1, t_2) = (1/\bar{\omega})\exp(-\zeta\omega_0(t_1+t_2))\sin\bar{\omega}(t_2-t_1) \tag{17a}$$

and the components of the initial state $X(t_0)$ in equation (14) can be expressed as

$$x_1(0) = \frac{x_1(t_1)\alpha_{12}(t_2) - x_1(t_2)\alpha_{12}(t_1)}{J(t_1, t_2)} \tag{17b}$$

and

$$x_2(0) = \dot{x}_1(0) = \frac{x_1(t_2)\alpha_{11}(t_1) - x_1(t_1)\alpha_{11}(t_2)}{J(t_1, t_2)} \tag{17c}$$

Observe that $J(t_1, t_2)$ vanishes for $\bar{\omega}(t_2 - t_1) = \pi$, but it is not identically zero for all t_1 and t_2 values. Thus according to Moyal's degree of randomness criteria the response process $\{x_1(t)\}$ has two degrees of randomness. To further illustrate the vanishing of the Jacobian for certain specific t values assume that the initial state $X(t_0)$ jointly belongs to a normal process with properties:

$$E[x_1(t_0)] = E[\dot{x}_1(t_0)] = 0 \tag{18a}$$

$$E[x_1(t_0)x_1(t_0)] = \sigma_1^2,\ E[\dot{x}_1(t_0)\dot{x}_1(t_0)] = \sigma_2^2 \tag{18b}$$

and

$$E[x_1(t_0)\dot{x}_1(t_0)] = r\sigma_1\sigma_2 \tag{18c}$$

The joint probability density function of the initial state can now be set in the form

$$f(X(t_0)) = \frac{1}{2\pi\sigma_1\sigma_1(1-r^2)^{\frac{1}{2}}} \exp\left[-\frac{1}{2(1-r^2)}\left\{\frac{x(t_0)^2}{\sigma_1^2} - \frac{2r\,x(t_0)\dot{x}(t_0)}{\sigma_1\sigma_2} + \frac{x(t_0)^2}{\sigma_2^2}\right\}\right] \tag{19}$$

Therefore from equation (14)

$$f(x_1(t_1),\quad x_1(t_2)) = f(X(t_0))\frac{1}{|J(t_1, t_2)|} \tag{20}$$

where $J(t_1, t_2)$ and the components of $X(t_0)$ are given by (17a), (17b) and (17c).

At $t_2 = t_1 + \pi/\bar{\omega}$, for any arbitrary constant C, one gets $x_1(t_2) = Cx(t_1)$, and $r = 1$. This case of perfect correlation with $r = 1$ and the existence of $f(x(t_1), x_1(t_2))$ at $t_2 = \pi/\bar{\omega} + t_1$ are not strictly compatible. Still it is possible to describe $\{x(t_1)\, x(t_1 + \pi/\bar{\omega})\}$ in terms of a degenerate normal process such that the corresponding joint density function is concentrated on the straight lines [20]

$$\frac{x_1(t_1)}{\sigma_3} = \frac{x_1(t_2)}{\sigma_4}, \quad t_2 = \pi/\bar{\omega} + t_1 \tag{21}$$

where $x_1(t_1)$ and $x_1(t_2)$ have the properties

$$E[x_1(t_1)x_1(t_1)] = \sigma_3^2, \quad E[x_1(t_2)x_1(t_2)] = \sigma_4^2$$

and

$$E[x_1(t_1)x_1(t_2)] = \rho\sigma_3\sigma_4$$

As expected, with $\rho \to 1$, the conditional probability density function

$$f(x_1(t_2)|x_1(t_1)) = \frac{1}{\sigma_4(2\pi(1-\rho^2))^{\frac{1}{2}}} \exp\left[-\frac{\{x_1(t_2) - (\sigma_4/\sigma_3)\rho x_1(t_1)\}^2}{2\sigma_4^2(1-\rho^2)}\right] \tag{22}$$

approaches the dirac delta function $\delta(x_1(t_2) - (\sigma_4/\sigma_3)\,\rho x_1(t_1))$, which is consistent with (21). Therefore at $t_2 = t_1 + \bar{\omega}/\pi$ one gets

$$f(x_1(t_1),\quad x_2(t_2),\quad t_2 - t_1 = \pi/\bar{\omega}) = f(x_1(t_1))\,\delta(x_1(t_2) - \frac{\sigma_4}{\sigma_3}x_1(t_1)) \tag{23}$$

where the singularity line masses indicate the deterministic functional dependence between $x_1(t_1)$ and $x_1(t_1 + \bar{\omega}/\pi)$.

3. Application to a Helicopter Blade Motion

Consider the flapping or the out of plane motion of a lifting rotor blade with a single degree of freedom. It is assumed that the blade is rigid against twisting and bending and the flapping hinge is centrally located. At small advance ratios, say up to 0.6, the linearized homogeneous part of the equation in a rotating frame of reference has the state matrix [5]

$$F(t)=\begin{bmatrix}0 & 1\\ -\left(1+\frac{\gamma\mu}{6}\cos t\right) & -\left(\gamma/8+\frac{\gamma\mu}{6}\sin t\right)\end{bmatrix},\quad 0<t\leqq T \tag{24}$$

where the parameters γ and μ are respectively the Lock inertia number and the advance ratio. It will be assumed that the joint density of the initial state is given. To be somewhat in the range of current helicopter flight regimes consider two typical cases, i) $\mu=0$, $\gamma=4$; and ii) $\mu=0.3$ and $\gamma=4$. The first case refers to hovering and the second one for forward flight.

In the perturbation scheme one considers the perturbed state matrix

$$F(t)=F_0+\varepsilon F_1(t) \tag{25}$$

where F_0 is given by (16a) with $\omega_0=1$ and $2\omega_0\zeta=0.5$, and

$$F_1(t)=\begin{bmatrix}0 & 0\\ -d\cos t & -d\sin t\end{bmatrix},\quad d=\frac{\gamma\mu}{6} \tag{26}$$

Assume the state transition matrix in the form $\Phi(t,0)+\varepsilon\Phi(t,0)$ where the perturbation parameter ε is only a mathematical artifice, the solution of interest being at $\varepsilon=1$. Observe that the solution of the zero-order equation

$$\dot{\Phi}_0(t,0)=F_0\Phi_0(t,0)\,,\quad \Phi(0,0)=I_2 \tag{27}$$

is given by (16b) and (16c). After some algebra the solution of the first-order perturbation equation

$$\dot{\Phi}_1(t,0)=F_0\Phi_1(t,0)+(F_1(t)\Phi_0(t,0))\,,\quad \Phi_1(0,0)=0 \tag{28}$$

can be set in the form

$$\beta_{11}(t)=(d/2\bar{\omega})(\exp(x\zeta\omega_0 t))[(\omega_0^2/\bar{\omega})A(t)-B(t)-(\zeta\omega_0/\bar{\omega})C(t)] \tag{29a}$$

$$\beta_{12}(t)=(d/2\bar{\omega})(\exp(-\zeta\omega_0 t))[D(t)+(\zeta\omega_0/\bar{\omega})(t)-(1/\bar{\omega})C(t)] \tag{29b}$$

$$\beta_{21}(t)=\dot{\beta}_{11}(t)\quad\text{and}\quad \beta_{22}(t)=\dot{\beta}_{12}(t) \tag{29c}$$

where

$$A(t)=(\bar{\omega}/(2\bar{\omega}+1))\cos(1+\bar{\omega})t+(\bar{\omega}/(2\bar{\omega}-1))\cos(\bar{\omega}-1)t-(4\bar{\omega}^2/(4\bar{\omega}^2-1))\cos\bar{\omega}t \tag{30a}$$

$$B(t)=(\bar{\omega}/(2\bar{\omega}-1))\cos(\omega-1)t-(\bar{\omega}/2\bar{\omega}+1)\cos(1+\bar{\omega})t-(2\bar{\omega}/(4\bar{\omega}^2-1))\cos\bar{\omega}t \tag{30b}$$

$$C(t)=(-\bar{\omega}/(2\bar{\omega}+1))\sin(1+\bar{\omega})t-(\omega/(2\bar{\omega}-1))\sin(1-\bar{\omega})t+(2\bar{\omega}/(4\bar{\omega}^2-1))\sin\bar{\omega}t \tag{30c}$$

$$D(t)=(-\bar{\omega}/(2\bar{\omega}-1)\sin(1-\bar{\omega})t+(\bar{\omega}/(1+2\bar{\omega}))\sin(1+\bar{\omega})t+(2-4\bar{\omega}^2)/(4\bar{\omega}^2-1)\sin\bar{\omega}t \tag{30d}$$

Now the state transition matrix is given by

$$\Phi(t,0)=\Phi_0(t,0)+\Phi_1(t,0)=\begin{bmatrix}\alpha_{11}+\beta_{11} & \alpha_{12}+\beta_{12}\\ \alpha_{21}+\beta_{21} & \alpha_{22}+\beta_{22}\end{bmatrix} \tag{31}$$

where equations (16b), (16c), (29a), (29b) and (29c) define the four elements of $\Phi(t,0)$. For the joint probability density function of the final state, the initial state components $x(0)$ and $\dot{x}(0)$ have to be expressed in terms of the final state components. According to (12b)

$$x(O)=\frac{x(t)\phi_{22}(t)-x(t)\phi_{12}(t)}{J(t)} \tag{32a}$$

$$\dot{x}(O)=\frac{\dot{x}(t)\phi_{11}(t)-x(t)\phi_{21}(t)}{J(t)} \tag{32b}$$

where, from (12a), the Jacobian $J(t)$ simplifies to

$$J(t) = \exp(ct + d \sin t) \tag{32c}$$

Similarly for the joint probability density function of the response random variables $x(t_1)$ and $x(t_2)$, one gets

$$x(O) = \frac{x(t_1)\,\phi_{12}(t_2) - x(t_2)\,\phi_{12}(t_1)}{J(t_1, t_2)} \tag{33a}$$

$$\dot{x}(O) = \frac{x(t_2)\,\phi_{11}(t_1) - x(t_1)\,\phi_{11}(t_2)}{J(t_1, t_2)} \tag{33b}$$

and from (15)

$$J(t_1, t_2) = \begin{vmatrix} \phi_{11}(t_1) & \phi_{12}(t_1) \\ \phi_{11}(t_2) & \phi_{12}(t_2) \end{vmatrix} \tag{33c}$$

Thus equation (32) and (33) respectively define the joint density of the final state components $x(t)$ and $\dot{x}(t)$ and of response random variables $x(t_1)$ and $x(t_2)$.

Now coming to numerical results, figure 1 refers to $J(t_i, t_2)$ values with $t_i = 1$ and 6. The full lines refer to the constant parameter or the hovering case and dotted curves for the time variable system or the forward flight case. The perturbation solutions for $d = 0.2$ ($\gamma = 4$, $\mu = 0.3$) have been verified with a direct numerical approach and the solutions from these two methods agree almost up to two significant figures. Observe that $J(t_1, t_2)$ in some sense indicates the range and the variation in the stochastic structure of the transient response. Numerical study has indicated that for $t \simeq 5\pi$ the transient effect is negligible and the response with deterministic excitations will be essentially deterministic.

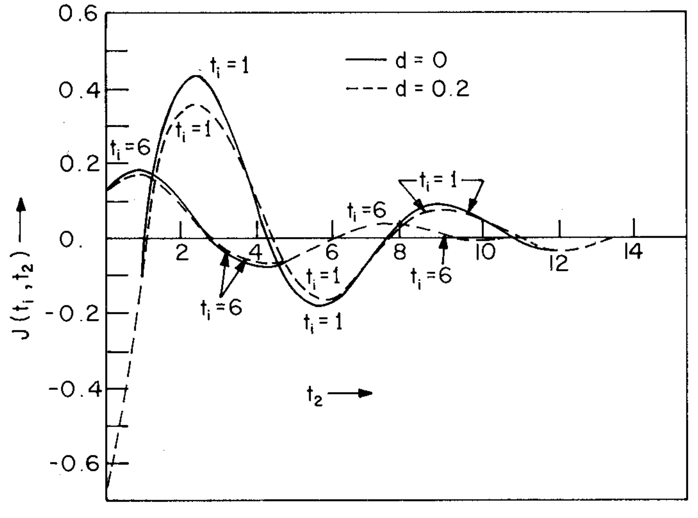

Figure 1. Jacobian of the response random variables.

4. Concluding Remarks

With the perturbation scheme as presented here it is possible to treat certain time variable cryptodeterministic systems of engineering interest. The method in general provides a computationally convenient formulation for the joint probability density function of the final state and of response random variables, including certain degenerate cases with singularity line masses in the joint density functions. In conformity with the degree of randomness criteria the method provides a complete stochastic characterization of the response process.

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